Lecture 2: The Solow-Swan Growth Model

Lei Pan
School of Accounting, Economics and Finance
Curtin University

Outline

- Some facts of growth
- Central questions of growth theory
- The Solow-Swan growth model
  - Environment of the model economy
  - Equilibrium
  - Steady state or stationary equilibrium
  - Comparative static analysis
  - Golden rule
- Evaluation of the Solow-Swan model
- Empirical application-growth accounting
- Further reading: "Explaining cross-country income differences"

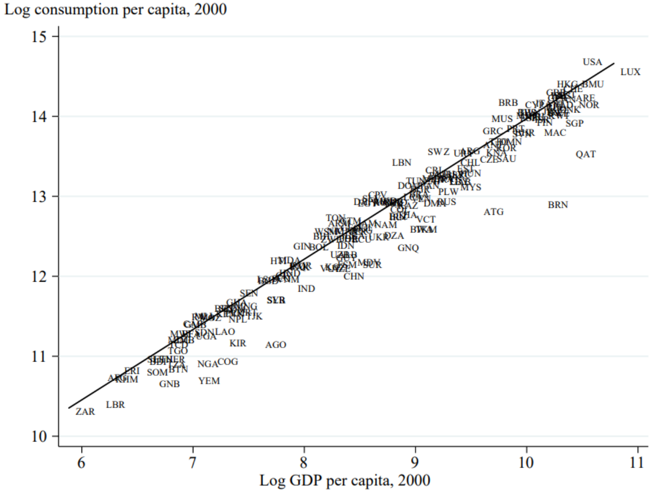
What is Economic Growth?

- Standard economists' answer: Growth in GDP per capita (or per worker), Y/L
- Does growth in per capita GDP reflect anything important about the improvement of living standards?
  - No consideration, for example, of the distribution of income within the economy (inequality)

Measures of Living Standards

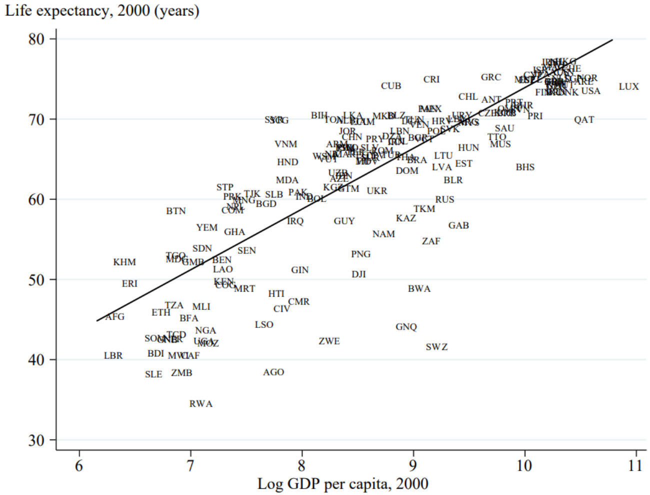
- Normally, the focus on two statistics of the average person's well-being: income and GDP per worker (productivity measure), and income and GDP per capita (welfare measure).
- They correlate with many other important statistics measuring well-being: infant mortality, life expectancy, consumption, etc.

Consumption per capita vs. GDP per capita



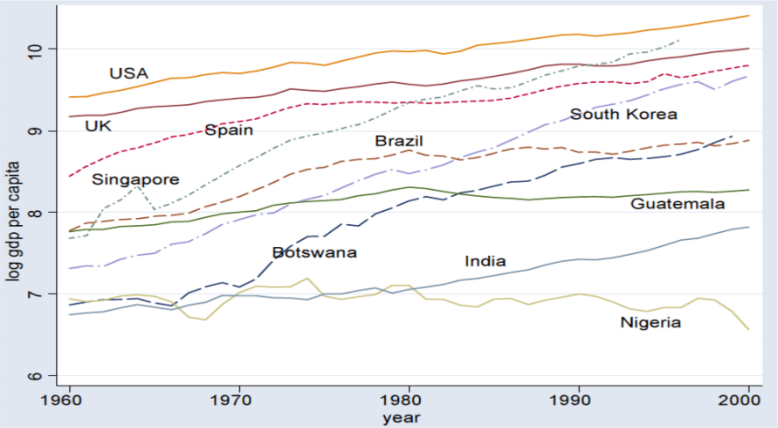
Source: Acemoglu (2008). Introduction to Modern Economic Growth

Life expectancy vs. GDP per capita



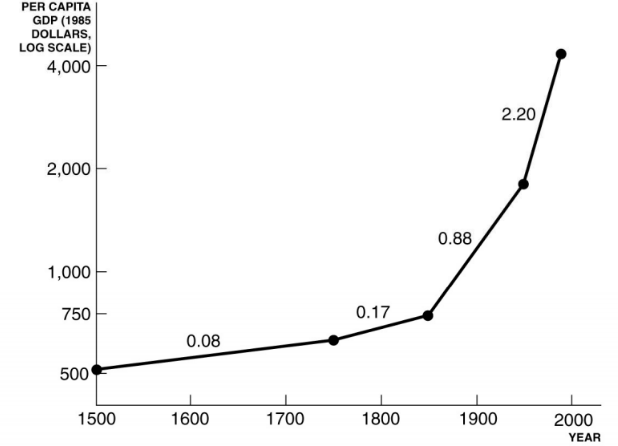
Source: Acemoglu (2008). Introduction to Modern Economic Growth

Economic Growth and Income Differences



Source: Acemoglu (2008). Introduction to Modern Economic Growth

World per capita GDP and Growth Rates



Source: Charles Jones. Introduction to Economic Growth.

The Power of Growth Rates (1 of 2)

- Time to double (rule of 70): Assume that y_t grows at a constant rate g. Then

$$y_{t+1} = (1+g)y_t \quad \text{and} \quad y_{t+k} = (1+g)^k y_t, k \geq 0$$

- What is the time needed for y_t to double?
  - We want to solve for k^* that satisfies: $2y_t = (1+g)^{k^*} y_t$
  - Taking natural logs from both sides gives:

$$ln2 = ln(1+g)k^*, \quad \text{or} \quad k^* = \frac{ln2}{g} \approx \frac{0.7}{g} = \frac{70}{100g}$$

where $ln(1+g) = g$ using Maclaurin expansion (see Appendix) of $ln(1+x)$, and g is expressed in percentage terms

The Power of Growth Rates (2 of 2)

- Thus, if $g = 0.02$ (e.g. U.S.), GDP per capita will double every $70/2 = 35$ years.
- If $g = 0.06$ (e.g. South Korea) ≈ every 12 years.
- If, e.g. the difference in age between grandparents and grandchildren is about 48 years, Korean (future) grandchildren of the current generation will be about $2^4 = 16$ times wealthier than the current generation.

A summary of stylized facts of growth (1 of 2)

- **Fact 1.** There is enormous variation in incomes per capita across countries. The poorest countries have per capita incomes less than 5% of per capita incomes in the richest countries.
- **Fact 2.** Rates of economic growth vary substantially across countries.
- **Fact 3.** Growth rates are not generally constant over time. For the world as a whole, growth rates were close to zero over most of history but have increased sharply in the 20-th century.
  - The same applies to individual countries. Countries can move from being "poor" to being "rich" (e.g. South Korea), and vice versa (e.g. Argentina).

A summary of stylized facts of growth (2 of 2)

For the U.S. over the last century:

- The real interest rate shows no trend, up or down.
- The share of labour and capital costs in income, although fluctuating, have no trend.
- The average growth rate in output per capita has been relatively constant over time, i.e. the U.S. is on a path of sustained growth of incomes per capita.
- These are called **Kaldor's stylized facts of growth**.

Central questions of growth theory

- What determines growth? In particular,
  - How much capital accumulation accounts for growth?
  - How important is technological progress?
- Why do countries grow at different rates?
- What causes growth rates to decline?
- In this lecture we use simple Neoclassical growth model developed by Robert Solow (1956) and Trevor Swan (1956) independently.

Solow-Swan Model (1 of 2)

- A major paradigm:
  - widely used in policy making
  - benchmark against which most recent growth theories are compared
- Looks at the determinants of economic growth and the standard of living in the long run.

Solow-Swan Model (2 of 2)

- A starting point for more complex models.
- Abstracts from modeling heterogeneous households (in tastes, abilities, etc.), heterogeneous sectors in the economy, and social interactions. It is a one-good economy with simplified individual decisions.
- We'll discuss the Solow-Swan model in discrete time.

Environment of the model economy

- **Time** is discrete, and time horizon is infinite, $t = 0, 1, ...$ (days/weeks/months/years)
- **Firm**: produce output with a constant-return-to-scale (CRS) production function:

$$Y_t = F(K_t, A_t L_t) \tag{1}$$

where A_t denotes the labour augmenting technology. Assume exogenous technological growth: A_t grows at rate g, i.e.

$$A_{t+1} = (1 + g)A_t \tag{2}$$

Capital stock is accumulated through investment:

$$K_{t+1} = (1 - \delta)K_t + I_t \tag{3}$$

Capital depreciates at the rate $\delta \in (0, 1)$ (machines utilised in production lose some of their value due to wear and tear). 1 unit now "becomes" $1 - \delta$ units next period.

Assumption: Inada Conditions

- We'll assume that F satisfies Inada conditions:

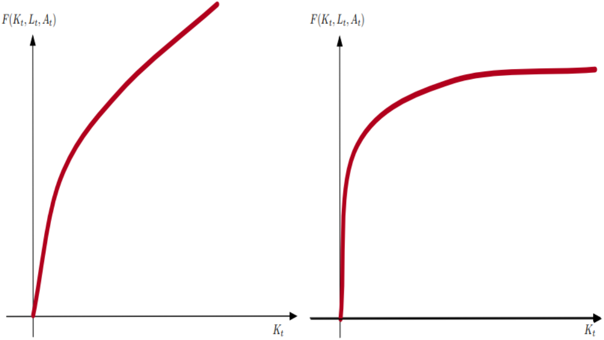
$$\lim_{L\to 0} F_L(K, AL) = \infty, \text{ and } \lim_{L\to\infty} F_L(K, AL) = 0 \text{ for all } K > 0, \text{ all } A$$
$$\lim_{K\to 0} F_K(K, AL) = \infty, \text{ and } \lim_{K\to\infty} F_K(K, AL) = 0 \text{ for all } L > 0, \text{ all } A$$

- Capital is essential in production (can be relaxed):

$$F(0, AL) = 0 \text{ for all } L \text{ and } A$$

- The conditions say that the "first" units of labour and capital are highly productive, and that when labour and capital are sufficiently abundant, the use of a marginal unit does not add anything to the existing output.

Inada Conditions



Production function on the left does NOT satisfy the Inada conditions, the marginal product of capital does not decline to zero as capital is accumulated. The production function on the right does satisfy the Inada conditions.

Consumers

- Consumers supply labour to firms, consume and save. But consumers' behavior is much simplified in this model.
- Assume that there is an exogenous population growth such that the stock of labour grows at rate n, i.e.

$$L_{t+1} = (1+n)L_t \tag{4}$$

- Assume that aggregate consumption is constant fraction of aggregate output (a bit unrealistic), i.e.

$$C_t = (1-s)Y_t \tag{5}$$

  where $s \in (0,1)$ is a parameter. Then it follows that the total savings of consumers is given by:

$$S_t = sY_t \tag{6}$$

- **Initial conditions**: The initial stock of capital, labour, and technology are given, denoted by K_0, L_0 and A_0 respectively.

Equilibrium (Solution of the Model)

- **Question**: What are we solving for? What does the solution look like?
  - Given initial conditions, the solution needs to determine all endogenous variables in every period.
  - Typically, the solution is characterized by a few equations, known as **equilibrium conditions**.
  - We will show that for this simple model the equilibrium condition is a single equation that describes how capital (key endogenous variable of the model) evolve over time!
- **Question**: Where to start?

Market Clearing Condition

- Gross saving equals gross investment

$$S_t = I_t = Y_t - C_t \tag{7}$$

- Plugging (7) into the capital accumulation equation, can get:

$$K_{t+1} = (1 - \delta)K_t + S_t$$

- Recall from (6) and (1)

$$S_t = sY_t \text{ and } Y_t = F(K_t, A_t L_t)$$

  Then gives the law of motion

$$K_{t+1} = (1 - \delta)K_t + sF(K_t, A_t L_t) \tag{8}$$

Intensive Form

- We typically rewrite Eq.(8) in its intensive form. Define $k_t = \frac{K_t}{A_t L_t}$ as capital per unit of effective labour. Dividing both sides of Eq.(8) by $A_t L_t$:
  - Left hand side:

$$\frac{K_{t+1}}{A_t L_t} = \frac{K_{t+1}}{A_{t+1} L_{t+1}} \frac{A_{t+1} L_{t+1}}{A_t L_t} = (1+g)(1+n)k_{t+1}$$

  - Right hand side:

$$\frac{F(K_t, A_t L_t)}{A_t L_t} = F(\frac{K_t}{A_t L_t}, 1) = F(k_t, 1) \equiv f(k_t) \tag{9}$$

  We call

$$y_t = f(k_t)$$

  as the intensive form of the production function, where $y_t \equiv \frac{Y_t}{A_t L_t}$ is the output per unit of effective labour.

Transition Equation

- Then we can obtain the transition equation of capital per unit of effective labour:

$$k_{t+1} = \frac{(1-\delta)}{(1+n)(1+g)} k_t + \frac{s}{(1+n)(1+g)} f(k_t) \equiv g(k_t) \tag{10}$$

- This transition equation characterizes the equilibrium/solution of the model.
- Given an initial condition K_0 and exogenous sequences $\{A_t, L_t\}$ we generate an endogenous sequence $\{K_t\}$ by iterating on the transition equation.
- Once we know K_t, we also know output and consumption etc.

$$Y_t = F(K_t, A_t L_t)$$
$$C_t = (1-s) F(K_t, A_t L_t)$$

- In short, k_t (with exogenous A_t, L_t) summarises equilibrium dynamics of the economy.

Steady State (1 of 2)

- **Question**: What is the long run behavior of the model? What does the long run equilibrium look like?
- Does k_t converge to a level k^* such that it will stay there forever? If such a k^* exists, then the economic system has a steady state.
- How to find k^*? Let all k terms in the transition equation equal to k^*:

$$k^* = g(k^*) \tag{11}$$

$$k^* = \frac{(1-\delta)}{(1+n)(1+g)}k^* + \frac{s}{(1+n)(1+g)}f(k^*)$$

$$[(1+n)(1+g) - (1-\delta)]k^* = sf(k^*)$$

$$(n+g+gn+\delta)k^* = sf(k^*)$$

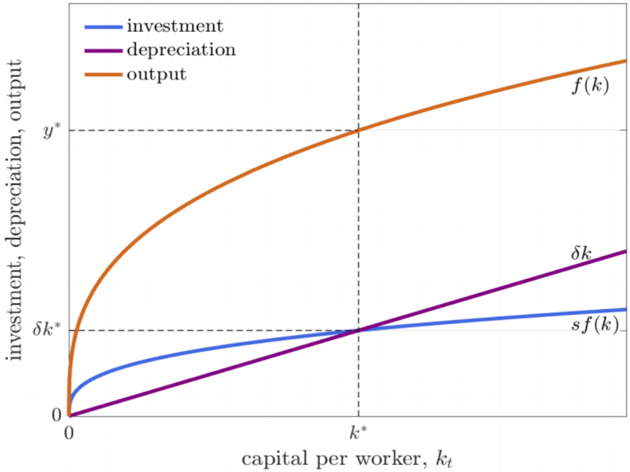
Steady State (2 of 2)

- Since gn is small, we ignore it, then we have:

$$(n + g + \delta)k^* = sf(k^*) \tag{12}$$

- Eq.(12) indicates that the steady state level of capital per unit of effective labour k^* is the intersection of the curve $sf(k)$ and the line $(n + g + \delta)k$.
- Notice that $sf(k)$ is actual investment per unit of effective labour, while $(n + g + \delta)k$ is the amount of investment per unit of effective labour that is needed to keep k and its existing level – "break even" investment. Therefore, Eq.(12) states that the actual investment must equal the break-even investment in a steady state.
- When k converges to k^*, the model reaches its steady state. Aggregate capital, output, consumption, and investment all grow at constant rate. Such a situation is called a balanced growth path.

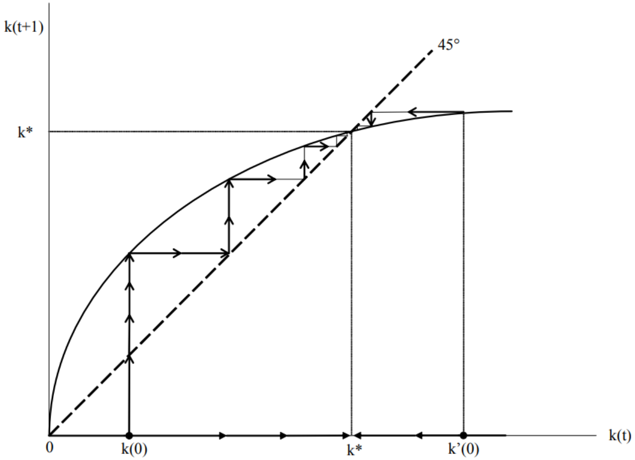
Solow Diagram (with constant A and L)



Balanced Growth Path

- Features of the balanced growth path:
  - By assumption, L_t and A_t are growing at rates n and g, respectively.
  - $K_t = A_t L_t k^*$ is growing at $(n + g)$
  - $Y_t = A_t L_t f(k^*)$ is growing at $(n + g)$
  - $I_t = sY_t$ and $C_t = (1 - s)Y_t$ are both growing at $(n + g)$
  - Output per worker Y_t/L_t and capital per worker K_t/L_t are growing at the same rate g
  - The capital output ratio K_t/Y_t is a constant
- Convergence to the balanced growth path
  - Regardless of its starting point, k_t converges to a unique k^*. So the Solow-Swan economy is globally stable.
  - Starting from $k_0 < k^*$, the convergence to k^* is faster when k is lower, i.e., the convergence is faster when k_t is further away from k^*.

Transition Dynamics



Comparative Static Analysis

- After equilibrium is solved, a typical further analysis is to examine how endogenous variables respond to variations in the parameters of the model. For example, how endogenous variable change in response to
  - changes in the population growth rate, n
  - changes in the technological growth rate, g
  - changes in the saving rate, s
  - changes in the depreciation rate, δ

Golden Rule (1 of 3)

- **Question**: Is the equilibrium a good outcome?
- Standard of assessment here: Golden rule allocation. Is k^* the golden-rule level?
- The concept of "golden rule" was first stated by Phelps (AER 1961, 1965). It proposed a way to evaluate whether the economy save too much or too little, or equivalently whether the economy has too much or too little capital.
- The golden- rule level of capital stock is defined as the one that maximises steady state consumption.
- In the context of Solow-Swan model, the Golden rule allocation represents a steady State or a balanced growth path that yields the maximum sustainable level of consumption.

Golden Rule (2 of 3)

- The golden rule level of k, denoted by k^*_{GR}, maximises the steady state consumption per unit of effective labour

$$c = f(k) - i = f(k) - (n + g + \delta)k$$

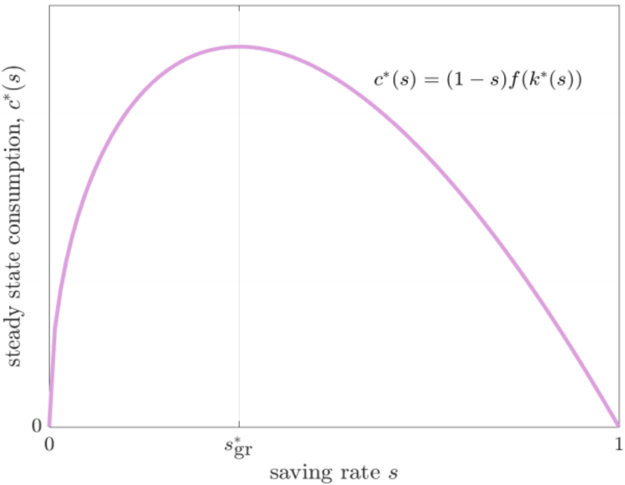
Therefore, k^*_{GR} is determined by:

$$f'(k^*_{GR}) = (n + g + \delta) \tag{13}$$

Dynamic Inefficiency

- When s is such that $k^* < k^*_{GR}$, an increase in s will increase steady state consumption per worker.
- When s is such that $k^* > k^*_{GR}$, an increase in s will decrease steady state consumption per worker.
- In latter case, steady-state consumption can be increased by saving less.
- In a sense, there is "too much capital", a form of "dynamic inefficiency".
- A more general way to evaluate the efficiency of an equilibrium outcome is too see whether the equilibrium allocation is Pareto-efficient or Pareto-optimal. We'll discuss this in details in the Diamond model.

Golden Rule (3 of 3)



Evaluations of the Solow-Swan Model (1 of 4)

- Are the model's equilibrium properties consistent with stylized facts of growth?
  - Features of the balanced growth path of the model are reasonably consistent with the Kaldor's growth facts.
  - Some testable comparative static properties are broadly consistent with the data: Output per worker is positively related to the saving rate s and negatively related to the rate of population growth n.
- What does the model say about the central questions of growth theory?
  - The model identifies two possible sources of variation in Y/L: differences in K/L and differences in A:

$$Y/L = F(K/L, A)$$

- The model predicts that only growth in A can lead to permanent growth in output per worker.

Evaluations of the Solow-Swan Model (2 of 4)

- The model also predicts that variations in the accumulation of physical capital do not account for significant part of either worldwide economic growth or cross-country income differences. Consider a Cobb-Douglas production function:

$$\frac{Y}{L} = (\frac{K}{L})^{\alpha} A^{1-\alpha}$$

  where α is estimated to be roughly $1/3$ in most countries. Output per worker in major industrialised countries today is 10 times larger than it was 100 years ago, and 10 times larger than it is in poor countries. To account for this on the basis of differences in capital, capital per worker must differ by a factor of 1000. However,there is no evidence of such differences in capital stocks.

Evaluations of the Solow-Swan Model (3 of 4)

- Can the model explain income differences across countries?
  - According to the model, income differences across countries are partly due to differences in s and n across countries.
  - Using a Cobb-Douglas production function, k^* can be solved as $k^* = (\frac{s}{n+g+\delta})^{\frac{1}{1-\alpha}}$, then along balanced growth path:

$$\frac{Y}{L} = Af(k^*) = A{k^*}^{\alpha} = A(\frac{s}{n+g+\delta})^{\frac{\alpha}{1-\alpha}}$$

  - Suppose economics have different s and n. If economics have conditionally converged to their balanced growth paths, the output per worker in country i relative to country j should exhibit the following relationship:

$$\frac{Y_i/L_i}{Y_j/L_j} = (\frac{s_i}{s_j} \times \frac{n_j+g+\delta}{n_i+g+\delta})^{\frac{\alpha}{1-\alpha}}$$

Evaluations of the Solow-Swan Model (4 of 4)

- Note that this relationship is testable with data. Such a test is known as conditional convergence test.
  - The data are inconsistent with this hypothesis: The exogenous growth model leads to an over prediction of GDP per capita relative to the United States.
  - If countries have not conditionally converged yet, countries with lower relative GDP per capita (relative to the predicted level) would exhibit more rapid growth. There is evidence that economies are converging conditionally.

Figure 7.3: Actual GDP per capita and predicted GDP per capita (relative to the United States) conditional on saving rate and rate of population growth

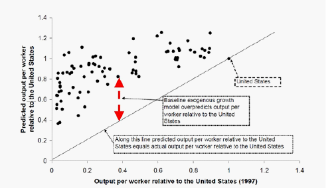
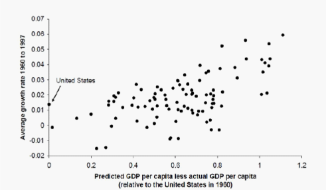


Figure 7.4: Growth rate of GDP per capita plotted against predicted GDP per capita less actual GDP per capita (relative to the United States) in 1960



Empirical Application: Growth Accounting (1 of 2)

- **Question**: How much of output growth over some period is due to increases in various factors of production, and how much stems from other forces?
- Growth accounting, pioneered by Abramovitz (1956) and Solow (1957), provides a way of dealing with this subject.
- Consider $Y(t) = F(K(t), A(t), L(t))$. This implies:

$$\frac{dY(t)}{dt} = \frac{\partial Y(t)}{\partial K(t)}\frac{dK(t)}{dt} + \frac{\partial Y(t)}{\partial L(t)}\frac{dL(t)}{dt} + \frac{\partial Y(t)}{\partial A(t)}\frac{dA(t)}{dt}$$

- Denote $\frac{dX(t)}{dt} \equiv \dot{X}(t)$, dividing both sides by $Y(t)$ and rewriting the right-hand side yields:

$$\frac{\dot{Y}(t)}{Y(t)} = [\frac{K(t)\partial Y(t)}{Y(t)\partial K(t)}]\frac{\dot{K}(t)}{K(t)} + [\frac{L(t)\partial Y(t)}{Y(t)\partial L(t)}]\frac{\dot{L}(t)}{L(t)} + [\frac{A(t)\partial Y(t)}{Y(t)\partial A(t)}]\frac{\dot{A}(t)}{A(t)}$$

Empirical Application: Growth Accounting (2 of 2)

$$\equiv \alpha_K(t)\frac{\dot{K}(t)}{K(t)} + (1 - \alpha_K(t))\frac{\dot{L}(t)}{L(t)} + R(t) \tag{14}$$

- See an example (with specific form of production function) of derivation in Appendix.
- The Solow residual, $R(t)$, is generally viewed as a measure of productivity growth in particular, a measure of TFP (total factor productivity) growth.
- The Growth rates of Y, K and L can be directly measured. If capital and labour earn their marginal products, α_K and α_L can be measured using data on the shares of factor payments. $R(t)$ can then be measured as the residual in Eq.(14).

Appendix: Maclaurin Expansion of $ln(1+x)$

- The Maclaurin expansion of a function $f(x)$ is:

$$f(x) = f(0) + f'(0)x + f''(0)x^2 + ...$$

- For $ln(1+x)$, we know $f(0) = ln(1+0) = ln(1) = 0$
- Then $f'(x) = \frac{dln(1+x)}{dx} = \frac{1}{1+x}$, thus, $f'(0) = \frac{1}{1+0} = 1$
- Hence, $ln(1+x) = 0 + x + ... = x$

Appendix: Deriving the Growth Accounting Equation (1 of 2)

Consider a Cobb-Douglas CRS production function:

$$Y = AK^{\alpha}L^{1-\alpha}$$

Take total differential of production function:

$$dY = K^{\alpha}L^{1-\alpha}dA + \alpha AK^{\alpha-1}L^{1-\alpha}dK + (1-\alpha)AK^{\alpha}L^{-\alpha}dL$$

Note: $\frac{Y}{A} = K^{\alpha}L^{1-\alpha}$

$$dY = \frac{Y}{A}dA + \alpha AK^{\alpha-1}L^{1-\alpha}dK + (1-\alpha)AK^{\alpha}L^{-\alpha}dL$$

Appendix: Deriving the Growth Accounting Equation (2 of 2)

$$dY = Y\frac{dA}{A} + \alpha AK^{\alpha-1}L^{1-\alpha}dK + (1-\alpha)AK^{\alpha}L^{-\alpha}dL$$

$$dY = Y\frac{dA}{A} + \alpha AK^{\alpha-1}L^{1-\alpha}K\frac{dK}{K} + (1-\alpha)AK^{\alpha}L^{-\alpha}dL$$

$$dY = Y\frac{dA}{A} + \alpha AK^{\alpha}L^{1-\alpha}\frac{dK}{K} + (1-\alpha)AK^{\alpha}L^{-\alpha}dL$$

$$dY = Y\frac{dA}{A} + \alpha Y\frac{dK}{K} + (1-\alpha)AK^{\alpha}L^{-\alpha}L\frac{dL}{L}$$

$$dY = Y\frac{dA}{A} + \alpha Y\frac{dK}{K} + (1-\alpha)AK^{\alpha}L^{1-\alpha}\frac{dL}{L}$$

$$dY = Y\frac{dA}{A} + \alpha Y\frac{dK}{K} + (1-\alpha)Y\frac{dL}{L}$$

$$\frac{dY}{Y} = \frac{dA}{A} + \alpha\frac{dK}{K} + (1-\alpha)\frac{dL}{L}$$